ON THE SOLUTION OF THE STOKES PROBLEM FOR AN EQUIPOTENTIAL SURFACE SPECIFIED IN THE FORM OF A SPHEROID

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The solution of the Stokes problem is used to obtain explicit expressions in the form of special rapidly convergent series for the projections of the intensity of the Earth's regularized gravitational field at its surface and outside it.

1. We introduce the right-handed orthogonal coordinate system O_1xyz, placing its origin at the center of the Earth, directing the z-axis along the vector u of the Earth's angular rotational velocity, and locating its x-axis at the intersection of the equatorial and Greenwich meridian planes.

By F_x, F_y, F_z we denote the projections on the axes xyz of the intensity of the Earth's regularized gravitational field at some point O with the coordinates x, y, z. The solution of the Stokes problem for an equipotential surface specified in the form of a spheroid (Clairaut ellipsoid) yields the following Expressions [1 to 3] for these projections at the point O situated outside the ellipsoid:

$$F_x = -Px + C\frac{\partial K}{\partial x}, \quad F_y = -Py + C\frac{\partial K}{\partial y}, \quad F_z = -Qz + C\frac{\partial K}{\partial z} \qquad (1.1)$$

In these expressions

$$P = 2\pi D\frac{a^2b}{(a^2-b^2)^{3/2}}\left(\tan^{-1} l' - \frac{l'}{1+l'^2}\right)$$

$$Q = 4\pi D\frac{a^2b}{(a^2-b^2)^{3/2}}(l' - \tan^{-1} l'), \qquad K = 2\pi D\frac{a^2b}{\sqrt{a^2-b^2}}\tan^{-1} l' \qquad (1.2)$$

where a and b are the major and minor semiaxes of the Clairaut ellipsoid; l' is the second eccentricity of the confocal ellipsoid passing through the point O; D and C are constants.

The quantity D is found from the condition [1]

$$\frac{u^2}{2\pi D} = \frac{3+l^2}{l^3}\tan^{-1} l - \frac{3}{l^2} \qquad (1.3)$$

where l is the second eccentricity of the Clairaut ellipsoid. The constant C can be obtained by comparing the acceleration due to gravity as obtained from Formulas (1.1) to (1.3) with its measured value g_e at sea level on the equator. For determining C we have Expression

$$g_e = 2\pi DCa + P_0 a - u^2 a \tag{1.4}$$

where P_0 is the value of P on the surface of the Clairaut ellipsoid.

2. Let us specify the position of the point O in the coordinate system O_1xyz in terms of the geocentric coordinates: the distance r form the Earth's center, the latitude φ and longitude λ. Next, we introduce the associated trihedron $Ox_1y_1z_1$ of the geocentric coordinate grid, directing its z_1-axis along $r = O_1O$ and pointing the y_1-axis northward in the plane of the meridian. Then, from (1.1) and (1.2) and from the definition of the confocal ellipsoid we have

$$F_{x_1} = 0, \quad F_{y_1} = 2\pi D \sin\varphi \cos\varphi\,(R - CS), \quad F_{z_1} = -2\pi D\,(T + R\cos^2\varphi + CU) \tag{2.1}$$

In these expressions

$$R = \frac{a^2br}{(a^2-b^2)^{3/2}}\left(3 \tan^{-1} l' - \frac{l'}{1+l'^2} - 2l'\right), \qquad T = \frac{2a^2br}{(a^2-b^2)^{3/2}}\,(l' - \tan^{-1} l')$$

$$S = \frac{a^4be^2}{r}\,\frac{(b^2+\nu)^{1/2}}{(b^2+\nu)^2\cos^2\varphi + (a^2+\nu)^2\sin^2\varphi} \tag{2.2}$$

$$U = \frac{a^2b}{r}\,\frac{[(b^2+\nu)\cos^2\varphi + (a^2+\nu)\sin^2\varphi]\,(b^2+\nu)^{1/2}}{(b^2+\nu)^2\cos^2\varphi + (a^2+\nu)^2\sin^2\varphi}$$

where e is the first eccentricity of the equipotential ellipsoid and ν is the parameter of the confocal ellipsoid $(l' = [(a^2-b^2)/(b^2+\nu)]^{1/2})$.

The right sides of Formulas (2.1) can be expanded in series in powers of e.

From the equation of the confocal ellipsoid and from the definition of its second eccentricity we have

$$l' = \frac{\sqrt{2\,(a^2-b^2)}}{\sqrt{r^2 - a^2 + b^2 + \sqrt{r^4 + (a^2-b^2)^2 - 2r^2(a^2-b^2)\cos 2\varphi}}} \tag{2.3}$$

Hence,

$$l' = \frac{ae}{r}\,\mu, \qquad \mu = 1 + \frac{1}{2}\left(\frac{ea}{r}\right)^2\cos^2\varphi + \frac{1}{16}\left(\frac{ea}{r}\right)^4\left(\frac{3}{8}\cos^2\varphi - \frac{7}{32}\sin^2 2\varphi\right) + \ldots \tag{2.4}$$

Now

$$R = be^2\left(\frac{a}{r}\right)^4\mu^5\left(-\frac{2}{5} + \frac{4}{7}l'^2 + \ldots\right), \qquad T = b\left(\frac{a}{r}\right)^2\mu^3\left(\frac{2}{3} - \frac{2}{5}l'^2\,\frac{1}{7}l'^4 + \ldots\right) \tag{2.5}$$

or, substituting in l' and μ from (2.4),

$$R = be^2\left(\frac{a}{r}\right)^4\left[-\frac{2}{5} + \left(\frac{ae}{r}\right)^2\left(\frac{4}{7} - \cos^2\varphi\right) + \ldots\right] \tag{2.6}$$

$$T = b\left(\frac{a}{r}\right)^2\left[\frac{2}{3} + \left(\frac{ae}{r}\right)^2\left(\cos^2\varphi - \frac{2}{5}\right) + \left(\frac{ae}{r}\right)^4\left(\frac{2}{7} + \frac{1}{4}\cos^2\varphi - \frac{9}{32}\sin^2 2\varphi\right) + \ldots\right]$$

Similarly,

$$\nu = r^2\left[1 - \left(\frac{a}{r}\right)^2 - \left(\frac{ae}{r}\right)^2\sin^2\varphi + \frac{1}{4}\left(\frac{ae}{r}\right)^4\sin^2 2\varphi + \ldots\right] \tag{2.7}$$

so that

$$S = Cbe^2\left(\frac{a}{r}\right)^4\left[1 + \left(\frac{ea}{r}\right)^2\left(\frac{7}{2}\cos^2\varphi - 2\right) + \ldots\right] \tag{2.8}$$

$$U = Cb\left(\frac{a}{r}\right)^2\left[1 + \left(\frac{ae}{r}\right)^2\left(\frac{3}{2}\cos^2\varphi - 1\right) + \left(\frac{ae}{r}\right)^4\left(1 - \frac{5}{8}\cos^2\varphi - \frac{35}{32}\sin^2 2\varphi\right) + \ldots\right]$$

Substituting (2.6) and (2.8) into (2.1), we have Formulas

$$F_{y_1} = \pi D b e^2 \left(\frac{a}{r}\right)^4 \sin 2\varphi \left\{-\frac{2}{5} - C + \left[-3\left(\frac{1}{7} + \frac{C}{2}\right) + \sin^2\varphi\left(1 + \frac{7}{2}C\right)\right]\left(\frac{ae}{r}\right)^2 + \ldots\right\} \tag{2.9}$$

$$F_{z_1} = -2\pi D b \left(\frac{a}{r}\right)^2 \left\{\frac{2}{3} + C + \left[\frac{1}{5} + \frac{1}{2}C - \sin^2\varphi\left(\frac{3}{5} + \frac{3}{2}C\right)\right]\left(\frac{ae}{r}\right)^2 + \left[\frac{3}{28} + \frac{3}{8}C + \sin^2\varphi\left(\frac{5}{28} + \frac{5}{8}C\right) + \sin^2 2\varphi\left(-\frac{5}{16} - \frac{35}{32}C\right)\right]\left(\frac{ae}{r}\right)^4 + \ldots\right\}$$

which are the expansions we were seeking. From (2.3) we conclude that series (2.4) converge absolutely and uniformly. This implies the convergence of series (2.6),(2.8) and hence of (2.9). The rapid convergence of series (2.9) is guaranteed by the smallness of ae/r.

Let us compute the consonants D and C. From (1.3) we have

$$D = \frac{u^2}{2\pi e^2}\,\frac{15}{4}\left(1 - \frac{1}{7}e^2 + \frac{1}{49}e^4 + \ldots\right) \tag{2.10}$$

The constant C appears in (2.9) as part of the product πDC. From (1.1),(1.2) and (1.4) we have

$$\pi DC = \frac{g_e}{2a} + \frac{u^2}{2e^2}\left(-\frac{5}{2} + \frac{13}{7}e^2 + \frac{8}{49}e^4 + \ldots\right) \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), multiplying out the series, and carrying out some obvious regroupings, we arrive at the Expressions

$$F_{y_1} = \frac{g_e}{2}(q - e^2)\left(\frac{a}{r}\right)^4 \sin 2\varphi\left[1 + \frac{e^2(7e^2 - 30q)}{14(q - e^2)}\right]\left\{1 + \left[\frac{30q - 21e^2}{14(q - e^2)} + \frac{7e^2 - 10q}{2(q - e^2)}\sin^2\varphi\right]\left(\frac{ae}{r}\right)^2 + \ldots\right\} \tag{2.12}$$

$$F_{z_1} = -g_e\left(\frac{a}{r}\right)^2\left\{1 - \frac{e^2}{2} - \frac{e^4}{8} + q\left(\frac{3}{2} - \frac{15}{28}e^2\right) + \left[\frac{1}{2}e^2 - \frac{1}{4}e^4 + q\left(-\frac{1}{2} + \frac{15}{14}e^2\right) - \sin^2\varphi\left(\frac{3}{2}e^2 - \frac{3}{4}e^4 + q\left(-\frac{3}{2} + \frac{45}{14}e^2\right)\right)\right]\left(\frac{a}{r}\right)^2 + \left[\frac{3}{8}e^2 - \frac{15}{28}q + \sin^2\varphi\left(\frac{5}{8}e^2 - \frac{25}{28}q\right) - \sin^2 2\varphi\left(\frac{35}{32}e^2 - \frac{25}{16}q\right)\right]e^2\left(\frac{a}{r}\right)^4 + \ldots\right\}$$

in which q denotes the ratio of the centrifugal acceleration to the acceleration due to gravity on the equator, $q = u^2 a/g_e$. The quantity q is of the same order of smallness as e^2.

Let us specify the values of the coefficients appearing in Formulas (2.12). We take (*)

$$g_e = 978.049 \text{ см/сек}^2,\quad u = 7.29212 \text{ сек}^{-1},\quad a = 6378250 \text{ м}$$

Then

$$e^2 = 0.00669342,\quad q = 0.00346775$$

*) The values of a, e^2, q are taken for the parameters of Krasovskii's ellipsoid [4].

$$g_e / 2\,(q - e^2) = -1.577, \quad 1 + e^2 (7e^2 - 30\,q) / 14\,(q - e^2) = 1.008$$

$$e^2 (30q - 21\,e^2) / 14\,(q - e^2) = 0.005$$

$$e^2 (7e^2 - 10\,q) / 2\,(q - e^2) = -0.013, \quad 1 - {}^1/_2\, e^2 - {}^1/_8 e^4 + q\,({}^3/_2 - {}^{15}/_{28}\, e^2) = 1.001837$$

$${}^1/_2 e^2 - {}^1/_4 e^4 + q\,(-{}^1/_2 + {}^{15}/_{14}\, e^2) = 0.001627, \qquad ({}^3/_8\, e^2 - {}^{15}/_{28}\, q)\, e^2 = 0.000005$$

$${}^3/_2 e^2 - {}^3/_4 e^4 + q\,(-{}^3/_2 + {}^{45}/_{14}\, e^2) = 0.004879, \qquad ({}^5/_8\, e^2 - {}^{25}/_{28}\, q)\, e^2 = 0\,000007$$

$$({}^{35}/_{32}\, e^2 - {}^{25}/_{16}\, q)\, e^2 = 0.000013$$

To within 0.02 cm/sec², Formulas (2.12) can be replaced by the simpler ones

$$F_{y_1} = \frac{g_e}{2}(q - e^2)\left(\frac{a}{r}\right)^4 \sin 2\varphi \tag{2.13}$$

$$F_{z_1} = -g_e\left(\frac{a}{r}\right)^2\left[1 - \frac{e^2}{2} + \frac{3}{2}q + \frac{q - e^2}{2}(-1 + 3\sin^2\varphi)\left(\frac{a}{r}\right)^2\right]$$

$${}^1/_2 g_e (q - e^2) = -1.58, \quad {}^1/_2 (q - e^2) = 0.0016, \quad 1 - {}^1/_2 e^2 + {}^3/_2 q = 1.0019$$

3. We introduce the geographic coordinates of the point O : the latitude φ', longitude λ, and the distance h from the point O to the Clairaut ellipsoid along the normal to the latter. It is easy to obtain the relations

$$r^2 = a^2 + h^2 + 2ah\,\sqrt{1 - e^2\sin^2\varphi'} - \frac{a^2e^2(1 - e^2)\sin^2\varphi'}{1 - e^2\sin^2\varphi'} \tag{3.1}$$

$$\tan\varphi = \left[1 - \frac{ae^2}{a + h(1 - e^2\sin^2\varphi')^{1/2}}\right]\tan\varphi'$$

relating the coordinates r φ to h, φ'. Relations (3.1) make it possible to express the projections F_{y_1}, F_{z_1} in terms of h and φ'. Let us obtain these expressions, limiting ourselves to the case where the ratio h/a is of the same order of smallness as e^2.

From (3.1) we have

$$\left(\frac{a}{r}\right)^2 = 1 - \frac{2h}{a} + \frac{3h^2}{a^2} - \frac{3h}{a}e^2\sin^2\varphi' + e^2\sin^2\varphi' + e^4\left(\sin^2\varphi' - \frac{1}{2}\sin^2 2\varphi'\right) \tag{3.2}$$

$$\sin\varphi = \sin\varphi'(1 - e^2\cos^2\varphi')$$

We substitute (3.2) into (2.12). Multiplying out the series (we note that q and e^2 are of the same order of magnitude) and retaining only the terms of the order e^4, we obtain

$$F_{y_1} = \frac{g_e}{2}(q - e^2)\sin 2\varphi'\left[1 - 4\frac{h}{a} - \frac{qe^2}{q - e^2} - \frac{e^2(e^2 + 2q)}{2(q - e^2)}\sin^2\varphi'\right] \tag{3.3}$$

$$F_{z_1} = -g_e\Big[1 - \frac{e^2}{2}\sin^2\varphi' + q\left(1 + \frac{3}{2}\sin^2\varphi'\right) + e^4\left(-\frac{1}{8}\sin^2\varphi' - \frac{3}{32}\sin^2 2\varphi'\right) +$$
$$+ e^2q\left(-\frac{17}{28}\sin^2\varphi' + \frac{1}{16}\sin^2 2\varphi'\right) + \frac{h}{a}e^2(3\sin^2\varphi' - 1) -$$
$$- \frac{hq}{a}(1 + 6\sin^2\varphi') - \frac{2h}{a} + \frac{3h^2}{a^2}\Big]$$

Next, we introduce the associated trihedron $Ox_2y_2z_2$ of the geographic coordinate grid. Its axis z_2 is directed along the positive normal of the Clairaut ellipsoid; the axis y_2 points nothwards and lies in the plane of the geographic meridian. The relative position of the trihedra $Ox_1y_1z_1$ and $Ox_2y_2z_2$ is characterized by the angle $(\varphi' - \varphi)$, so that

$$F_{y_2} = F_{y_1}\cos(\varphi' - \varphi) - F_{z_1}\sin(\varphi' - \varphi), \qquad F_{z_2} = F_{y_1}\sin(\varphi' - \varphi) + F_{z_1}\cos(\varphi' - \varphi) \tag{3.4}$$

From the second formula of (3.1), we have to within e^4 that

$$\sin(\varphi' - \varphi) = e^2 \sin\varphi' \cos\varphi' \,(1 + e^2 \sin^2\varphi' - h/a) \tag{3.5}$$

$$\cos(\varphi' - \varphi) = 1 - {}^1\!/_2\, e^4 \sin^2\varphi' \cos^2\varphi'$$

Substituting (3.3) and (3.5) into (3.4), we arrive at the following expressions for F_{y_2}, F_{z_2} :

$$F_{y_2} = g_e \sin 2\varphi' \left[\frac{q}{2}\left(1 + \frac{e^2}{2}\sin^2\varphi'\right) + \frac{h}{a}\left(\frac{e^2}{2} - 2q\right)\right]$$

$$F_{z_2} = -g_e\left[1 - \frac{e^2}{2}\sin^2\varphi' + q\left(1 + \frac{3}{2}\sin^2\varphi'\right) + e^4\left(-\frac{1}{8}\sin^2\varphi' + \frac{1}{32}\sin^2 2\varphi'\right) + \right.$$
$$+ e^2 q\left(-\frac{17}{28}\sin^2\varphi' - \frac{3}{16}\sin^2 2\varphi'\right) + \frac{h}{a}e^2(3\sin^2\varphi' - 1) -$$
$$\left. - \frac{h}{a}q(1 + 6\sin^2\varphi') - \frac{2h}{a} + \frac{3h^2}{a^2}\right]$$

By setting $h = 0$ in these expressions, we obtain the formulas for the projections $F_{y_2}^{\circ}$, $F_{z_2}^{\circ}$ of the intensity of the Earth's regularized gravitational field at its surface (on the surface of the equipotential Clairaut ellipsoid). If we now add to $F_{y_2}^{\circ}$, $F_{z_2}^{\circ}$ the projections on the axes $x_2 y_2 z_2$ of the centrifugal acceleration due to the Earth's rotation, we see that the first sum vanishes, while the second leads to the familiar formula for the normal gravitational force in the Helmert-Cassinis form [1 and 5].

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